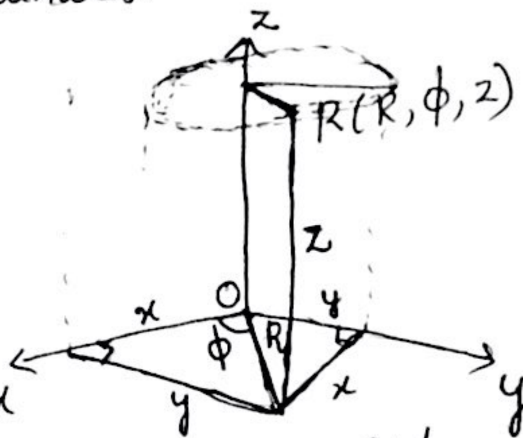
Change of variables in triple integrals:

1) Triple integral in cylindrical polar coordinates.

Here (x, y, z) are related three variables (R, ϕ, z) through the relations $x = R\cos\phi$, $y = R\sin\phi$, $z = z$. Then R, ϕ, z are called as cylindrical polar co-ordinates.

$$\iiint_R f(x, y, z)\, dx\, dy\, dz$$

$$= \iiint_R \psi(R, \phi, z)\, R\, dR\, d\phi\, dz$$



$\cos\phi = \frac{x}{R}$

$\sin\phi = y/R$

$z = z$

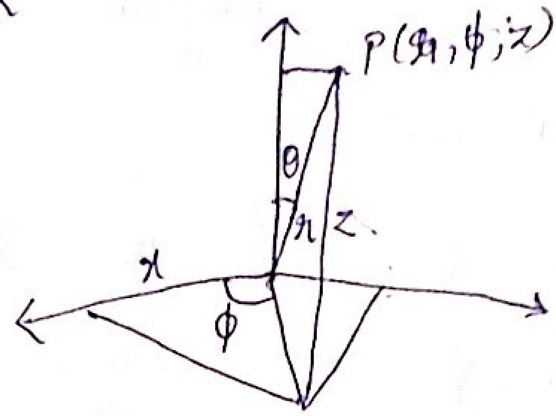
2) Triple Integral in spherical polar co-ordinates

Here (x, y, z) are related to three variables (r, θ, ϕ) through the relations.

$x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$.

Then (r, θ, ϕ) are called as spherical polar co-ordinates.

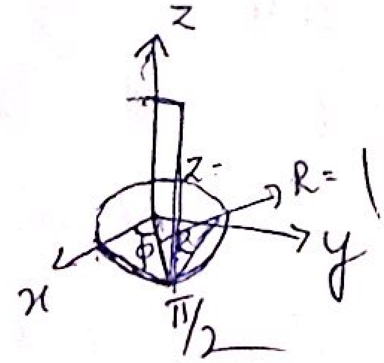
$$\iiint_R f(x,y,z)\,dx\,dy\,dz = \iiint_R \psi(r,\theta,\phi)\, r^2 \sin\theta \, dr\, d\theta\, d\phi$$



Problems:

1) If R is the region bounded by the planes $x=0$, $y=0$, $z=0$, $z=1$ and the cylinder $x^2+y^2=1$. Evaluate the integral $\iiint_R xyz\,dx\,dy\,dz$ by changing it to cylindrical polar co-ordinates.

Solution: Let (R,ϕ,z) be cylindrical polar co-ordinates. In the given region, R varies from 0 to 1, ϕ varies from 0 to $\pi/2$ and z varies from 0 to 1



$$\iiint_R xyz\,dx\,dy\,dz = \cancel{\iiint}$$

put $x = R\cos\phi$, $y = R\sin\phi$, $z = z$

$dx\,dy\,dz = R\,dR\,d\phi\,dz$

$$= \int_{R=0}^{1} \int_{\phi=0}^{\pi/2} \int_{z=0}^{1} (\underline{R} \cos\phi)(\underline{R}\sin\phi)\, z\, \underline{R}\, dR\, d\phi\, dz$$

$$= \int_{R=0}^{1} \int_{\phi=0}^{\pi/2} \left[\int_{z=0}^{1} R^3 \cos\phi \sin\phi\, z\, dR\, d\phi\, dz \right.$$

$$= \int_{R=0}^{1} \int_{\phi=0}^{\pi/2} \left[\int_{z=0}^{1} R^3 \cos\phi \sin\phi\, z\, dz \right] d\phi\, dR$$

$$= \int_{R=0}^{1} \int_{\phi=0}^{\pi/2} R^3 \cos\phi \sin\phi \left[\frac{z^2}{2} \right]_0^1 d\phi\, dR$$

$$= \int_{R=0}^{1} \int_{\phi=0}^{\pi/2} R^3 \cos\phi \sin\phi \, (1/2 - 0)\, d\phi\, dR$$

$$= 1/2 \int_{0}^{1} R^3 \left[\int_{0}^{\pi/2} \cos\phi \sin\phi\, d\phi \right] dR$$

$$= \frac{1}{2}\int_0^1 R^3 \int_0^{\pi/2} \frac{2\sin\phi\cos\phi}{2}\, d\phi \; dR$$

$$= \frac{1}{4}\int_0^1 R^3 \left[\int_0^{\pi/2} \sin 2\phi \; d\phi\right] dR$$

$$= \frac{1}{4}\int_0^1 R^3 \left[-\frac{\cos 2\phi}{2}\right]_0^{\pi/2} dR$$

$$= \frac{1}{4}\int_0^1 R^3 \left[\frac{-\cancel{\cos(2\pi/2)}^{\,-1}}{2} - \left(-\frac{\cos 0}{2}\right)\right] dR$$

$$= \frac{1}{4}\int_0^1 R^3 \left[-\frac{(-1)}{2} + \frac{1}{2}\right] dR$$

$$= \frac{1}{4}\int_0^1 R^3 \left(\frac{1}{2}+\frac{1}{2}\right) dR$$

$$= \frac{1}{4}\int_0^1 R^3 \, dR$$

$$= \frac{1}{4}\left[\frac{R^4}{4}\right]_0^1 = \frac{1}{4}\left[\frac{1}{4} - 0\right] = \frac{1}{16}$$

2) Evaluate $\iiint_R xyz\,dx\,dy\,dz$ over the positive octant of the sphere $x^2+y^2+z^2=a^2$ by transforming into cylindrical polar co-ordinates.

Solution: cylindrical polar co-ordinates (R, ϕ, z).

They are $x = R\cos\phi$, $y = R\sin\phi$, $z = z$

We have, $x^2+y^2+z^2 = a^2 \rightarrow (1)$

put $x = R\cos\phi$, $y = R\sin\phi$ in (1) we get $x^2+y^2+z^2 = a^2 \Rightarrow R^2+z^2 = a^2$

$\Rightarrow z = \sqrt{a^2-R^2}$

In the given region R varies from 0 to a, ϕ varies from 0 to $\pi/2$, z varies from 0 to $\sqrt{a^2-R^2}$, $dx\,dy\,dz = R\,dR\,d\phi\,dz$

$$\therefore \iiint_R xyz\,dx\,dy\,dz = \int_{\phi=0}^{\pi/2}\int_{R=0}^{a}\int_{z=0}^{\sqrt{a^2-R^2}} R\cos\phi\, R\sin\phi\, z\, R\,dR\,d\phi\,dz$$

$$= \int_{\phi=0}^{\pi/2}\int_{R=0}^{a}\left[\int_{z=0}^{\sqrt{a^2-R^2}} \left\{R^3\cos\phi\sin\phi\, z\right\} dz\right] dR\,d\phi$$

$$= \int_{\phi=0}^{\pi/2} \int_{R=0}^{a} R^3 \left[\cos\phi \sin\phi \; \frac{z^2}{2} \right]_0^{\sqrt{a^2-R^2}} dR \, d\phi$$

$$= \int_0^{\pi/2} \cos\phi \sin\phi \int_0^{a} R^3 \left[\frac{(\sqrt{a^2-R^2})^2}{2} - 0 \right] dR \, d\phi$$

$$= \int_0^{\pi/2} \cos\phi \sin\phi \left[\int_0^{a} R^3 \frac{(a^2-R^2)}{2} dR \right] d\phi$$

$$= \frac{1}{2} \int_0^{\pi/2} \cos\phi \sin\phi \left[\int_0^{a} (R^3 a^2 - R^5) \, dR \right] d\phi$$

$$= \frac{1}{2} \int_0^{\pi/2} \cos\phi \sin\phi \left[\frac{R^4}{4} a^2 - \frac{R^6}{6} \right]_0^{a} d\phi$$

$$= \frac{1}{2} \int_0^{\pi/2} \cos\phi \sin\phi \left[\frac{a^4 a^2}{4} - \frac{a^6}{6} \right] d\phi$$

$$= \frac{1}{2} \int_0^{\pi/2} \cos\phi \sin\phi \left(a^6/4 - \frac{a^6}{6} \right) d\phi$$

$$= \frac{1}{2} \int_0^{\pi/2} \cos\phi \sin\phi \, \frac{a^6}{12} \, d\phi$$

$$= \frac{a^6}{24} \int_0^{\pi/2} \frac{2 \sin\phi \cos\phi}{2} \, d\phi$$

$$= \frac{a^6}{48} \int_0^{\pi/2} \sin 2\phi \, d\phi$$

$$= \frac{a^6}{48} \left[-\frac{\cos 2\phi}{2} \right]_0^{\pi/2}$$

$$= \frac{a^6}{48} \left[-\frac{\cos(\cancel{2}\,\pi/\cancel{2})}{2} + \frac{\cos 0}{2} \right]$$

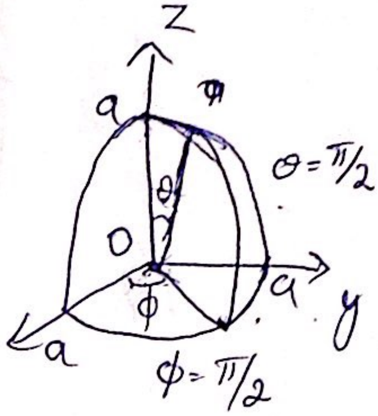
$$= \frac{a^6}{48} \left[-\frac{\overset{-1}{\cancel{\cos\pi}}}{2} + \frac{\overset{1}{\cancel{\cos 0}}}{2} \right]$$

$$= \frac{a^6}{48} \left[+\frac{1}{2} + \frac{1}{2} \right]$$

$$= a^6/48$$

3) Evaluate $\iiint_R xyz\,dx\,dy\,dz$ over the positive octant of the sphere $x^2+y^2+z^2=a^2$ by changing it to Spherical polar co-ordinates.

Solution: In the region, r varies from 0 to a, θ varies from 0 to $\pi/2$, ϕ varies from 0 to $\pi/2$



Spherical co-ordinates are,

$x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$

$dx\,dy\,dz = r^2\sin\theta\,dr\,d\theta\,d\phi.$

$$\iiint_R xyz\,dx\,dy\,dz = \int_{\theta=0}^{\pi/2}\int_{\phi=0}^{\pi/2}\int_{r=0}^{a} r\sin\theta\cos\phi\; r\sin\theta\sin\phi\; r\cos\theta\; r^2\sin\theta\,dr\,d\theta\,d\phi$$

$$= \int_{\theta=0}^{\pi/2}\int_{\phi=0}^{\pi/2}\left[\int_{r=0}^{a} r^5\sin^3\theta\cos\theta\cos\phi\sin\phi\,dr\right]d\phi\,d\theta$$

$$= \int_{\theta=0}^{\pi/2} \sin^3\theta \cos\theta \int_{\phi=0}^{\pi/2} \cos\phi \sin\phi \left[\int_{r=0}^{a} r^5 \, dr \right] d\phi \, d\theta$$

$$= \int_{\theta=0}^{\pi/2} \sin^3\theta \cos\theta \int_{\phi=0}^{\pi/2} \cos\phi \sin\phi \left[\frac{r^6}{6} \right]_0^a d\phi \, d\theta$$

$$= \int_{\theta=0}^{\pi/2} \sin^3\theta \cos\theta \int_{\phi=0}^{\pi/2} \cos\phi \sin\phi \left[\frac{a^6}{6} \right] d\phi \, d\theta$$

$$= \frac{a^6}{6} \int_{\theta=0}^{\pi/2} \sin^3\theta \cos\theta \left[\int_{\phi=0}^{\pi/2} \frac{2 \sin\phi \cos\phi}{2} \, d\phi \right] d\theta$$

$$= \frac{a^6}{12} \int_{\theta=0}^{\pi/2} \sin^3\theta \cos\theta \left[\int_{\phi=0}^{\pi/2} \sin 2\phi \, d\phi \right] d\theta$$

$$= \frac{a^6}{12} \int_{\theta=0}^{\pi/2} \sin^3\theta \cos\theta \left[-\frac{\cos 2\phi}{2} \right]_0^{\pi/2} d\theta$$

$$= \frac{a^6}{12} \int_0^{\pi/2} \sin^3\theta \cos\theta \left[-\frac{\cos \cancel{2}\cdot \pi/\cancel{2}}{2} + \frac{\cos 0}{2} \right] d\theta$$

$\cos\pi = -1$

$$= \frac{a^6}{12} \int_0^{\pi/2} \sin^3\theta \cos\theta \left(1/2 + 1/2 \right) d\theta$$

$$= \frac{a^6}{12} \int_0^{\pi/2} \sin^3\theta \cos\theta \, d\theta$$

put $\sin\theta = t$

$\cos\theta \, d\theta = dt$

$\theta = 0$

$t = 0$

$\theta = \pi/2$, $\sin\pi/2 = 1$

$t = 1$

$$= \frac{a^6}{12} \int_{t=0}^{1} t^3 \, dt$$

$$= \frac{a^6}{12} \left[\frac{t^4}{4} \right]_0^1$$

$$= \frac{a^6}{12} \left[\frac{1}{4} - 0 \right] = \frac{a^6}{12} \left(1/4 \right)$$

$$= \frac{a^6}{48}$$ //.

4) Evaluate $\iiint_R z(x^2+y^2)\,dx\,dy\,dz$, $x^2+y^2 \le 1$; $2 \le z \le 3$ by changing to cylindrical polar co-ordinates.

[Hint: $x^2+y^2 \le 1 \Rightarrow R^2 \le 1 \Rightarrow 0 \le R \le 1$

$0 \le \phi \le 2\pi$, $2 \le z \le 3$]   Ans: $\frac{5\pi}{4}$

5) Evaluate $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} \frac{dz\,dy\,dx}{\sqrt{1-x^2-y^2-z^2}}$ by converting to spherical polar co-ordinates.

[$z = 0$ to 1, $\theta = 0$ to $\pi/2$, $\phi = 0$ to $\pi/2$]

Ans: $\pi^2/8$